

Comment on “Mixing beliefs among interacting agents”

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Abstract

We comment on the derivation of the main equation in the bounded confidence model of opinion dynamics. In the original work, the equation is derived using an ad-hoc counting method. We point that the original derivation does contain some small mistake. The mistake does not have a large qualitative impact, but it reveals the danger of the ad-hoc counting method. We show how a more systematic approach, which we call micro to macro, can avoid such mistakes, without adding any significant complexity.

Keywords: *Opinion dynamics, bounded confidence, model, micro to macro, mean field.*

1 Introduction

One of the most famous mathematical models of opinion dynamics is presented in [5], in which the agents have continuous opinions and adjust them based on random encounters with other agents given that their opinions are below a fixed threshold d . Using a Taylor approximation and an ad-hoc counting method, the authors are able to derive an equation for the evolution of the distribution of opinions for small values of d .

We show that the original derivation does contain some small mistake. The mistake does not affect severely the model, as the qualitative behavior remains unchanged, but some values like the speed of convergence or the n -th order moments are impacted. This reveals the danger of the ad-hoc counting method.

We then show how the equation can be derived in a safer way, using a more systematic approach, which we call “micro to macro”. It consists in explicitly modelling the microscopic system before going to the fluid limit. The key technical tool is the drift equation. We illustrate that this is not more complex than the ad-hoc approach, and can be more easily validated.

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2 The Model

We consider a population of N peers, each of them having an opinion $x_i \in [0, 1]$ about some common subject. At every time step $k \in \mathbb{N}$, two peers are selected uniformly at random. If the distance between their opinions is sufficiently small, each of the opinions is modified, whereas if the opinions are far away they don't change. This is controlled with the uncertainty parameter d . Mathematically, we have that:

- If $|x_i - x_j| > d$ the opinions remain unchanged.
- If $|x_i - x_j| \leq d$ the opinions change the following way:

$$\begin{aligned} x_i(k+1) - x_i(k) &= \mu \cdot (x_j(k) - x_i(k)) \\ x_j(k+1) - x_j(k) &= \mu \cdot (x_i(k) - x_j(k)) \end{aligned}$$

where μ is a parameter (adaptation capacity) between 0 and $\frac{1}{2}$.

3 The Ad-Hoc Counting Method in [5, 10]

The final result is an equation for $\rho(t, x)$ defined as the PDF of opinions at time t :

$$\frac{\partial \rho(x, t)}{\partial t} = I_1 + I_2 \quad (1)$$

where I_1 and I_2 are described next. In [5] only the final result is stated without proof; the proof is available in [10], but in French only. Therefore, in the rest of this section we give a literal translation into English of the relevant parts of [10].

A simple enough calculation shows that, at the limit of the small values of d , the system has the tendency to amplify the irregularities of the population density according to their opinions, $\rho(x)$. During an elementary timestep corresponding to one interaction, the variation of $\rho(x)$ can be seen as the sum of two contributions:

A negative contribution corresponding to the probability for an agent of opinion x to interact and modify his opinion:

$$I_1 = -\rho(x) \int_{-d}^d \rho(x) \rho(x+y) dy \quad (2)$$

A positive contribution I_2 corresponding to the probability that an agent of an initially different opinion has opinion x after the interaction.

For this term, the obtaining of the normalizing constant is less trivial. The solution I have used is to consider a model in which the displacement of $x+y$ while interacting with $x+z$ is done around $\mu(y-z)$ with a gaussian probability. Letting the length of the gaussian go to zero, one gets our initial model

conveniently normalized. The positive term is therefore calculated by the double integral:

$$\begin{aligned}
I_2 &= + \int dy \cdot \rho(x+y) \int dz \cdot \rho(x+z) \delta((x+y) + \mu((x+z) - (x+y)) = x) \\
&= + \int dy \cdot \rho(x+y) \int dz \cdot \rho(x+z) \delta(y + \mu(z-y)) \\
&= + \int dy \cdot \rho(x+y) \int dz \cdot \rho(x+z) \frac{1}{\mu} \delta(z + \frac{1-\mu}{\mu}y) \\
&= \frac{1}{\mu} \int_{-\mu d}^{\mu d} \rho(x+y) \cdot \rho(x + \frac{\mu-1}{\mu}y) dy \tag{3}
\end{aligned}$$

For small d , one can replace in the integrals the terms in ρ by their Taylor series around x (limiting to second order in y):

$$\begin{aligned}
\delta\rho &= - \int_{-d}^d \rho \cdot \left(\rho + y\rho' + \frac{y^2}{2}\rho'' \right) dy \\
&+ \frac{1}{\mu} \int_{-\mu d}^{\mu d} \left(\rho + y\rho' + \frac{y^2}{2}\rho'' \right) \cdot \left(\rho + \frac{\mu-1}{\mu}y\rho' + \frac{1}{2} \left(\frac{\mu-1}{\mu}y \right)^2 \rho'' \right) dy \tag{4}
\end{aligned}$$

where all the terms in ρ are evaluated at x and $'$ represents the partial derivation with respect to x . Continuing the calculation, one obtains:

$$\begin{aligned}
\delta\rho &= -2d\rho^2 - \frac{d^3}{3}\rho'' \\
&+ \frac{1}{\mu}(2\mu d)\rho^2 + \frac{1}{\mu} \left[\frac{\mu-1}{\mu}\rho'^2 + \frac{1}{2} \left(\left(\frac{\mu-1}{\mu} \right)^2 + 1 \right) \rho\rho'' \right] \cdot 2 \cdot \frac{(\mu d)^3}{3} \tag{5}
\end{aligned}$$

After simplification, and supposing that the characteristic interaction time is τ , one finally finds at the first non-zero order:

$$\frac{\partial\rho}{\partial t} = \frac{d^3}{2\tau} \cdot \mu \cdot (\mu-1) \cdot \frac{\partial^2(\rho^2)}{\partial x^2} \tag{6}$$

There are three errors that alter the final result published in [5].

- (i) Equation (2) contains an extra term $\rho(x)$ and equation (5) lacks a factor ρ in the second term.
- (ii) In both I_1 and I_2 there is a multiplying factor 2 missing. Correcting both mistakes I_1 should read:

$$I_1 = -2\rho(x) \int_{-d}^d \rho(x+y) dy \tag{7}$$

and I_2 :

$$I_2 = \frac{2}{\mu} \int_{-\mu d}^{\mu d} \rho(x+y) \rho\left(x + \frac{\mu-1}{\mu}y\right) dy \quad (8)$$

$$\frac{\partial \rho}{\partial t} = \frac{2d^3}{3\tau} \cdot \mu \cdot (\mu-1) \cdot \frac{\partial^2(\rho^2)}{\partial x^2} \quad (9)$$

The error here is a division by $3/2$ instead of 2 . Note that the error is propagated to [12] and, to the best of our knowledge, has not been corrected anywhere else.

It is clear that the first correction is a typo, as it is corrected afterwards and the error is not propagated. Regarding correction (iii), it is a calculation mistake. Although the system's qualitative behavior isn't affected by this one, it might change the convergence speed of the system. Correction (ii) affects the time scale of the model by a factor of 2^1 . This is a common difficulty in the ad-hoc counting approach, which can be circumvented if, as we discuss next, we specify the microscopic model in detail.

The equation should read then, following the notation from [10]:

$$\frac{\partial \rho(x, t)}{\partial t} = \frac{2}{\mu} \int_{x-\mu d}^{x+\mu d} \rho\left(\frac{x - (1-\mu)y}{\mu}, t\right) \rho(y, t) dy - 2\rho(x, t) \int_{x-d}^{x+d} \rho(y, t) dy. \quad (10)$$

or, in a more compact (symmetric) way, as in a Boltzmann-like equation [8, 11], for $\mu \neq \frac{1}{2}$:

$$\begin{aligned} \frac{\partial \rho(x, t)}{\partial t} = & \frac{2}{2\mu-1} \int_{x-d(2\mu-1)}^{x+d(2\mu-1)} \rho\left(\frac{\mu x - (1-\mu)y}{2\mu-1}, t\right) \rho\left(\frac{\mu y - (1-\mu)x}{2\mu-1}, t\right) dy \\ & - 2\rho(x, t) \int_{x-d}^{x+d} \rho(y, t) dy \quad (11) \end{aligned}$$

Note that this mistake has also propagated to [1], in which it affects the calculation of the evolution of the second order moments, as it should read $M_2(t) = M_2(0)e^{-M_0 t}$ (there is a factor 2 in the exponent missing), and from there to several papers such as [4] or [2]. However, the derivation is also cited in [9], where it is partly corrected.

¹The system is described by “*A chaque pas de temps, deux agents sont tirés au hasard dans l'ensemble de la population*” (At every timestep, two agents are randomly chosen from the population), which differs from what is computed in the sequel: the variation of $\rho(x)$ should be calculated as the positive and negative contribution (which are correctly written) for the case when the peer with opinion x is the first element of the randomly chosen pair, and again, the same contribution (because of the symmetry of the problem) for the case when he is the second element of the randomly chosen pair.

4 An Alternative Method Based on Micro-to-Macro

In this section we describe how the evolution equation can be obtained in a safer way, using a micro to macro approach. The idea [3] is to first write the “drift equation” for the microscopic system, i.e. for the discrete time system defined in Section 2 where the number of peers is finite. Then, in a second step, the evolution equation for densities as in Eq. (10) is obtained in the limit of infinitely many peers.

In the microscopic system, the density is not a proper one, but a mixture of Dirac masses. Instead of manipulating densities, it is more convenient (and faster) to manipulate the integral of arbitrary test functions against the density. Therefore, we proceed as follows.

Step 1 (Micro) Let $M^N(t)$ be the occupancy measure of the microscopic system with N peers, defined as $M^N(t) = \frac{1}{N} \sum_{n=1}^N \delta_{x_n^N(t)}$. Let h be any bounded, measurable function defined on the space of one peer, i.e. on $[0, 1]$. The “drift” of the microscopic system is the operator \mathcal{G}^N defined by

$$\mathcal{G}^N(h)(\nu) \stackrel{\text{def}}{=} \mathbb{E} (h(M^N(t+1)) - h(M^N(t)) | M^N(t) = \nu)$$

where the starting condition ν must be of the form $\nu = \frac{1}{N} \sum_{n=1}^N \delta_{x_n}$. With some elementary manipulations we obtain:

$$\begin{aligned} \mathcal{G}^N(h)(\nu) &= \\ &= \frac{2}{N(N-1)} \frac{1}{N} \sum_{m < n} (h(\mu x_m + (1-\mu)x_n) + h(\mu x_n + (1-\mu)x_m) - h(x_m) - h(x_n)) 1_{\{|x_n - x_m| \leq d\}} \\ &= \frac{1}{N^2(N-1)} \sum_{m < n} (h(\mu x_m + (1-\mu)x_n) + h(\mu x_n + (1-\mu)x_m) - h(x_m) - h(x_n)) 1_{\{|x_n - x_m| \leq d\}} \\ &\quad + \frac{1}{N^2(N-1)} \sum_{n < m} (h(\mu x_m + (1-\mu)x_n) + h(\mu x_n + (1-\mu)x_m) - h(x_m) - h(x_n)) 1_{\{|x_n - x_m| \leq d\}} \\ &= \frac{1}{N^2(N-1)} \sum_{n, m} (h(\mu x_m + (1-\mu)x_n) + h(\mu x_n + (1-\mu)x_m) - h(x_m) - h(x_n)) 1_{\{|x_n - x_m| \leq d\}} \\ &= \frac{1}{N-1} \int_{[0,1]^2} (h(\mu y + (1-\mu)x) + h(\mu x + (1-\mu)y) - h(x) - h(y)) 1_{\{|x-y| \leq d\}} d\nu(x) d\nu(y) \\ &= \frac{2}{N-1} \int_{[0,1]^2} (h(\mu x + (1-\mu)y) - h(x)) 1_{\{|x-y| \leq d\}} d\nu(x) d\nu(y) \end{aligned} \tag{12}$$

Step 2 (Micro to Macro) Assume that when the number of peers N grows towards ∞ the occupancy measure M^N has a limit, say ν_t . We must also re-scale time by a factor of N , as we see next, so that $M^N(k) \approx \nu_{k/N}$. Whether this holds, and why, is an entirely different problem, which is not addressed in

[5, 10] and which do not address either here². Our purpose in this letter is solely to obtain a safe method for writing the equation. The method in e.g [3] says that, if we re-scale time by $1/N$, then the limit must satisfy

$$\frac{\partial \langle h, \nu_t \rangle}{\partial t} = \lim_{N \rightarrow \infty} N \mathcal{G}^N(h)(\nu_t)$$

if the limit exists. Intuitively, this is because in the limit we have $\langle h, \nu_{t+\frac{1}{N}} \rangle - \langle h, \nu_t \rangle \approx \mathcal{G}^N(h)(\nu_t)$. By Eq.(12), we obtain:

$$\frac{\partial \langle h, \nu_t \rangle}{\partial t} = 2 \int_{[0,1]^2} (h(\mu x + (1-\mu)y) - h(x)) 1_{\{|x-y| \leq d\}} d\nu(x) d\nu(y) \quad (13)$$

which is the main equation. To derive an equation for the density, we write $\nu_t(dx) = \rho(t, x)dx$, $\nu_t(dy) = \rho(t, y)dy$ and re-arrange Eq.(13) to obtain

$$\begin{aligned} \int_{[0,1]} h(x) \frac{\partial \rho(t, x)}{\partial t} dx &= 2 \int_{[0,1]} h(x) \left[\frac{1}{\mu} \int_{x-\mu d}^{x+\mu d} \rho\left(\frac{x - (1-\mu)y}{\mu}, t\right) \rho(y, t) dy \right. \\ &\quad \left. - \rho(x, t) \int_{x-d}^{x+d} \rho(y, t) dy \right] dx \end{aligned}$$

Eq.(12) follows by identification of $\frac{\partial \rho(t, x)}{\partial t}$ to the term between square brackets.

5 Conclusion

We pointed out that there were some mistakes in the derivation of Equation (6) in [5]. The mistakes do not affect the qualitative behavior of the system and were partly corrected in later works, however, their existence suggests that the ad-hoc counting method for deriving such equations is error prone. We described how a more systematic method can be used to avoid such pitfalls.

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²If the state space is replaced by a discrete approximation, this is a classical result of [6]; for the model in Section 2, this is proved in [7].

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